Let $(r_j)_{j\geq 1}$ be an enumeration of all rational #

in $\mathbb{R}$. Define the function φ to be

$$\varphi(x) = \sum_{\{j : r_j < x\}} \frac{1}{2^j}, \quad x \in \mathbb{R}.$$

Show that

(a) φ is strictly increasing.

(b) φ is discontinuous on $\mathbb{Q}$.

(c) φ is continuous on $\mathbb{R} \setminus \mathbb{Q}$.

Proof: If $y > x$, then

$$\varphi(y) = \sum_{r_j < y} \frac{1}{2^j} = \sum_{r_j < x} \frac{1}{2^j} + \sum_{x \leq r_j < y} \frac{1}{2^j} = \varphi(x) + \sum_{x \leq r_j < y} \frac{1}{2^j}.$$

Hence φ is increasing.

Suppose $x \in \mathbb{Q}$. Say $x = r_k$.

Then $\forall y > x$, we have

$$\varphi(y) - \varphi(x) = \sum_{x \leq r_j < y} \frac{1}{2^j} \geq \frac{1}{2^k} > 0.$$

Hence φ is discontinuous at x.

- Let $x \in \mathbb{R} \setminus \mathbb{Q}$.

Let $\varepsilon > 0$ & choose N s.t. $\frac{1}{2^N} < \varepsilon$.

Since $S_N := \{r_j : 1 \leq j \leq N\}$ is finite & $x \notin S_N$,

we can find $\delta := \delta_N > 0$ s.t. $(x-\delta, x+\delta) \cap S_N = \emptyset$.

Now, if $|y-x| < \delta$, then

$$
\begin{aligned}
|\varphi(y) - \varphi(x)| &\leq \sum_{r_j \in (x-\delta, x+\delta)} \frac{1}{2^j} \\
&= \sum_{\substack{r_j \in (x-\delta, x+\delta) \\ j > N}} \frac{1}{2^j} \\
&\leq \sum_{j > N} \frac{1}{2^j} \\
&= \frac{1}{2^N} < \varepsilon .
\end{aligned}
$$

Hence, φ is continuous at x.

$\square$

- Let $g: \mathbb{R} \to \mathbb{R}$ be a function satisfying

$$g(x+y) = g(x)g(y) \quad \forall x, y \in \mathbb{R}.$$

Show that if g is continuous at 0, then g is continuous at every point of $\mathbb{R}$.

Proof: Note that

$$g(x) - g(c) = g(c)\left(g(x-c) - g(0)\right) \quad \forall\, x, c \in \mathbb{R}.$$

Let $c \in \mathbb{R}$, and $(x_n) \to c$ be a seq in $\mathbb{R}$.

Then $\lim (x_n - c) = 0$.

By continuity at 0, $\lim\left(g(x_n - c)\right) = g(0)$.

Hence $\lim g(x_n) = g(0) + g(c) = g(c)$.

By sequence criterion for Continuity, g is continuous at c.

Continuous functions on Intervals

Let $f : [a,b] \to \mathbb{R}$ be a continuous function on a closed & bounded interval. Then:

controls the boundary behavior. eg.

① (Boundedness)

$\exists M \geq 0$ s.t. $|f(x)| \leq M \quad \forall x \in [a,b]$.

② (Max-Min thm)

$\exists x_1, x_2 \in [a,b]$ s.t. $f(x_1) = m$, $f(x_2) = M$ &

$$f(x_1) \leq f(x) \leq f(x_2) \quad \forall x \in [a,b].$$

③ (Intermediate Value Thm)

if $f(a) > k > f(b)$ for some $k \in \mathbb{R}$, then there exists some $\bar{x} \in [a,b]$ s.t. $f(\bar{x}) = k$.

Exercise 1: $f: \mathbb{R} \to \mathbb{R}$ be a continuous function.

Suppose $\lim_{x\to-\infty} f(x) = \lim_{x\to\infty} f(x) = L$.

Show that

(a) f is bounded on $\mathbb{R}$.

(b) f has either an absolute max. or min.

Proof: (a) By def., $\exists a<0, b>0$ s.t.

$|f(x)-L| \le 1$ whenever $x<a$ or $x>b$.

That is, $L-1 \le f(x) \le L+1$ for $x \in (-\infty, a) \cup (b, \infty)$.

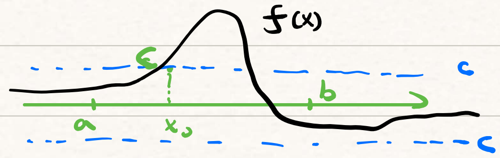
Since f is continuous on $[a,b]$, then by Boundedness Theorem, $|f(x)| \le M$ for some $M \ge 0$, whenever $x \in [a,b]$.

Hence $|f(x)| \le \max\{M, |L+1|, |L-1|\}$ on $\mathbb{R}$.

(b) Without loss of generality, assume $L=0$.

If f is constant, then $f \equiv L$ and f has both max & min.

If not, then there exists some $x_0 \in \mathbb{R}$ s.t. $f(x_0) \neq 0$.



Suppose $f(x_0) = c > 0$.

By def, there exist $a < x_0$ & $b > x_0$ s.t.

$$|f(x)| < c \text{ whenever } x < a \text{ or } x > b.$$

Since f is continuous on $[a,b]$, then by Max-Min Theorem,

there exist $x_1, x_2 \in [a,b]$ s.t.

$$f(x_1) \le f(x) \le f(x_2) \quad \forall x \in [a,b]. \quad \cdots ①$$

As $x_0 \in [a,b]$, we have $c = f(x_0) \le f(x_2)$.

Note that $\forall x \in (-\infty, a) \cup (b, \infty)$, we have

$$f(x) \le c \le f(x_2) \quad \cdots ②$$

Combining ① & ②, we conclude that

$$f(x) \le f(x_2) \quad \forall x \in \mathbb{R}.$$

Therefore, f has absolute max. on $\mathbb{R}$.

Similarly, f has absolute min on $\mathbb{R}$ if $f(x_0) = c < 0$.

□

Uniform Continuity

Definition: Let $A \subset \mathbb{R}$ & $f: A \to \mathbb{R}$ be a function.

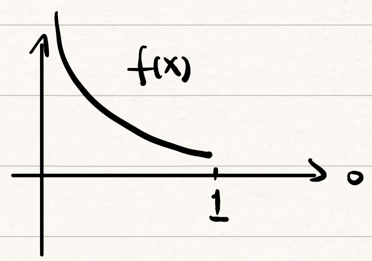
f is called uniformly continuous on A if $\forall \varepsilon > 0$,

$\exists \delta(\varepsilon) > 0$ s.t.

$$|f(x) - f(u)| < \varepsilon \quad \forall x, u \in A \ \& \ |x - u| < \delta.$$

- Try to compare the differene between uniform coutinuity & coutinuity

$f(x) = \frac{1}{x}$ on $(0, 1]$.



Ex 2: Prove $f(x)=\frac{1}{1+x^2}$ is uniformly continuous on $\mathbb{R}$.

Proof: Note that $\forall\, x, u \in \mathbb{R}$,

$$|f(x)-f(u)| = \left|\frac{1}{1+x^2}-\frac{1}{1+u^2}\right| = \underbrace{\frac{|x+u|}{(1+x^2)(1+u^2)}}_{\text{need to control this.}}|x-u|.$$

We have

$$\frac{|x+u|}{(1+x^2)(1+u^2)} \le \frac{|x|+|u|}{(1+x^2)(1+u^2)} \le \frac{|x|}{1+x^2}+\frac{|u|}{1+u^2} \le \frac{1}{2}+\frac{1}{2}=1.$$

Let $\varepsilon>0$ be arbitrary. Take $\delta=\varepsilon$.

Then whenever $x, u \in \mathbb{R}$ & $|x-u|<\delta$,

$$|f(x)-f(u)| \le |x-u| < \delta = \varepsilon.$$

By def, f is uniformly continuous on $\mathbb{R}$. $\square$

$$\text{uniform continuity} \xrightarrow{\qquad} \text{continuity}$$

NOT TRUE.

However, we have:

Theorem:

Let I be a closed & bounded interval. If $f: I \to \mathbb{R}$ is continuous on I, then f is uniformly continuous on I.

Exercise 3: Try to prove f in Ex 1 is uniformly cont.